Use of Variational Analysis in Aerospacecraft Design

JOHN A. THELANDER*
Douglas Aircraft Company Inc., Long Beach, Calif.

Useful procedures for the design of high-performance aerodynamic vehicles are outlined. Configuration variables, treated as special control variables, lead to development of corresponding influence-function relations. These relations utilize variational parameters and flight conditions from a reference optimum-flight path to 1) identify critical configuration (design) variables and 2) predict quantitative effects on the optimum performance (tradeoff factors) of configuration-variable perturbations. The procedures emphasize the use of simple methods to evaluate influence functions. Engineering estimates of configuration-variable effects on vehicle force characteristics (aerodynamic and propulsive) may be used for this evaluation. Examples use maximum-range and maximum-payload flight paths for ramjet-powered hypersonic aircraft to demonstrate application of the design procedures. General configuration variables, such as wing thickness, engine-inlet area, and thrust incidence, are used in these examples. Results show good agreement between the effects on performance estimated from the integrated influence function and the computed effect obtained from the corresponding optimum solutions.

Nomenclature

A_c = engine inlet capture area
C_D = aerodynamic drag coefficient, $C_D(M, \alpha) = \text{drag}/qS_w$
C_F = net-thrust coefficient, $C_F(M, \alpha) = \text{net thrust}/qA_c$
C_L = aerodynamic lift coefficient, $C_L(M, \alpha) = \text{lift}/qS_w$
$f_i(x, u, t)$ = state-variable derivatives, $f_i = \dot{x}_i$
g_0 = mass conversion factor, absolute to gravitational units
G = performance index, $G = G(t^i, t^f, x^i, x^f) = x_0(t^i)$
h = altitude
H = variational Hamiltonian, $H = \lambda^T\dot{x}$
I_{sp} = net-thrust specific impulse, $I_{sp}(M, \alpha)$
i_T = net-thrust incidence angle, air-breathing engine
J = final value of performance index, $J = x_0(t^f)$
m = vehicle mass
m_e = structure and fixed equipment mass
m_f = fuel mass
m_p = payload mass
M = Mach number
q = dynamic pressure
R_0 = earth radius
S_w = wing area
t = reference time (independent variable)
T_a = air-breathing-engine net thrust, $T_a(V, h, \alpha)$
u_j = general control-variable notation
V = velocity (relative to the earth)
x_i = general state-variable notation
X = range along surface of the earth
z_s = auxiliary variables associated with inequality constraints
α = angle of attack
γ = flight-path angle (relative to horizon)
η_s = inequality-constraint relation
θ = central range angle (geocentric)
λ_i = state adjoint variables (Lagrange multipliers)
λ_{uj} = control-variable variational parameter (influence function)
μ_s = inequality-constraint adjoint variables
ν_k = holonomic-constraint adjoint variables
φ_k = implicit-equality constraints (holonomic)
Ω = auxiliary-control-variable diagonal matrix, 2 diag (z_s)

Superscripts

i = initial value
f = final value

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* Senior Engineer, Aero Research Group. Member AIAA.

Introduction

As the envelope of sustained atmospheric flight is extended in velocity and altitude, the achievement of optimum vehicle design becomes increasingly difficult. Traditional quasi-steady performance methods are generally inadequate for comprehensive design analysis because predominantly dynamic flight paths are required to accomplish performance objectives efficiently. Credibility of traditional results decreases as optimum flight paths become more sensitive to design variables of interest.

For most surface vehicles, marine craft, and subsonic aircraft, the important design variables and performance tradeoff relations are well known. Hence, a design process based upon experience, using simple analytic and empirical relations, is adequate and reliable. This is not always true when innovational, high-performance, aerodynamic vehicles are to be designed.

Procedures demonstrated herein utilize data from optimum-flight-path solutions to generate useful design information for high-performance aerodynamic vehicles. Methods of variational calculus establish reference optimum flight paths. Variational parameters and flight conditions along a reference path are then used to compute design information. This information 1) identifies critical design variables and 2) evaluates the effects on performance of design-variable perturbations. The first phase of the analysis concerns development and evaluation of influence functions for particular design variables, and the latter phase is essentially the determination of tradeoff factors. Early review of this information focuses design effort on those configuration features (variables) that are most important.

The analysis and results shown are taken from Ref. 1. This work is a further application of the variational analysis presented by Leitmann[2] and is an extension of previous work by the author.[3] Although hypersonic aircraft are used for the demonstration examples, application to other classes of aircraft is straightforward.

Typically, design problems involve varied rules, conditions, and assumptions which are determined as a particular need (mission) is identified, compatible with the urgency, and the state of technology. The present analysis and examples assume that pertinent strategic, political, and economic decisions have been made, so that the design problem may be objectively defined.

Analysis

The development of influence functions and tradeoff relations associated with configuration variables in Ref. 1 is summarized here. A convenient starting point for this analysis is the fundamental variational relation as presented by Leitmann.[2] Briefly, the steps used to develop this relation are: 1) formulate the general perturbation analysis for a controllable system with constraints and identify the performance index, 2) imbed the unknown optimum solution in a parametric family of neighboring nonoptimum solutions which all satisfy the rules of motion for the system and the physical constraints on the system and problem, and 3) express the first-order change in the performance index that results from following a neighboring solution. The resulting equation relates change in performance to variations in 1) terminal-state values, 2) conditions at a corner, and 3) control-variable schedules along the path. Specifically, the terminal variation of the performance index $dx_0(t^f)$ is

$$dx_0(t^f) = dx_0(t^i) + [\lambda^T dx - H dt]_{t^i}^{t^f} + [H(t_c{}^+) - H(t_c{}^-)]dt_c - [\lambda^T(t_c{}^+) - \lambda^T(t_c{}^-)]\, dx(t_c) - \int_{t^i}^{c_-{}'} Q dt - \int_{c_+{}''}^{t^f} Q dt \quad (1)$$

where

$$Q = [\lambda^T(\partial f/\partial u) + \nu^T(\partial\varphi/\partial u) + \mu^T(\partial\eta/\partial u)]\delta u + \mu^T\Omega\delta z \quad (2)$$

The state variable x_0 is the performance index. In particular, $x_0{}^i = G(x^i, x^f, t^i, t^f)$ and $\dot{x}_0 = 0$, so that $x_0{}^f = x_0{}^i$. For convenience, initial and final values of x_0 are denoted by the symbols G and J, respectively.[2] Thus, $dx_0(t^i) = dG$ and $dx_0(t^f) = dJ$. (In the subsequent development, J identifies the total performance index and G refers to the component or analytical form of the index.) At a corner, the special limits c' and c'' were used so that dt_c may be positive or negative ($c' < t_c < c''$).

Using Eq. (1), the conditions necessary for zero change in the performance index, for each independent variation affecting the problem, provide optimum solution criteria.[2] The transversality condition relates terminal-state variations to performance and comprises the first two terms of Eq. (1). Corner conditions follow from the third and fourth terms in Eq. (1). The integrand terms, Eq. (2), lead to the Euler-Lagrange equations associated with control variables u_j and auxiliary variables z_s. These conditions, along with the Euler-Lagrange equations for the state variables x_i and the "maximum" principle, are the necessary variational conditions to be satisfied by optimum solutions.[2,3]

Configuration Variables

In the development of influence functions for design purposes, it is convenient to extend the scope of control variables to include design or configuration variables. Primary control variables u_j; $j = 1, 2, \ldots, n$; are unchanged. They are generally free and vary continuously (piecewise at least) along the flight path. Primary control variables include angle of attack, engine throttle setting, rocket-thrust deflection, and the like. Configuration variables u_j; $j = n + 1, n + 2, \ldots$; are treated as control variables in the problem formulation; however, during flight they generally remain fixed. Certain configuration variables may be adjusted at a discrete point, e.g., variable wing sweep. The number of configuration variables may be increased indefinitely to include any geometric and design parameters that can be quantitatively related to aerodynamic and propulsive forces. Typical configuration variables include wing area, thrust incidence, leading-edge sweep, engine compression ratio, inlet area, and fuselage fineness ratio. Figuratively, a vehicle may be viewed as a plastic device that retains its shape along a reference optimum flight path. Subsequent deformation to improve performance is guided by information derived from the reference solution.†

Reference Solution

A solution is first obtained for the design mission by using the variational analysis and performance optimization techniques already developed.[3] The configuration used for this reference solution is normally the most appropriate one available at the start of the design process. Such a configuration may be a modification of a precedent design or a preliminary estimate of the general features required. For the flight-path optimization, the requisite configuration data include aerodynamic and propulsive-system characteristics, as functions of Mach number, angle of attack, and throttle, together with appropriate geometric and vehicle-mass reference values.

Performance Tradeoffs

Incremental optimum performance due to a configuration-variable perturbation may be evaluated with Eq. (1) using data (multiplier, state-variable, and control variable histories) from the reference solution and known or estimated effects of the perturbation on configuration characteristics: aerodynamic, propulsive, and structural. Such increments supercede those ordinarily obtained by evaluating configuration adjustments along flight schedules prescribed by local optimization criteria.

Required physical conditions and necessary variational conditions[3] are determined for the design mission and satisfied by the reference solution. Primary control schedules are thus determined so that necessary conditions (transversality, corner criteria, Euler-Lagrange equations, and the maximum principle), are satisfied, together with prescribed physical constraints. Variation of the performance index $dx_0(t^f) = dJ$ is then zero for the reference solution. At this point, freedom to perturb configuration variables is assumed, and the question is posed as to what the effects on performance will be. The consequent effect of a specific control-variable perturbation, δu_b, will be denoted by $dJ(u_b)$.

It should be emphasized that the detailed expression for the performance index G includes effects of configuration variables on fixed structure or equipment. Typically, the performance index for a maximum payload mission is $m_p = J \equiv G = m^f - m_e$. Here, m^f is the vehicle total mass at the final condition, m_p is the payload mass, and m_e is the combined mass of structure, fixed equipment, crew, reserve fuel, etc.‡ For many configuration-variable perturbations, it is likely that vehicle structure and equipment will be affected. Thus, the detail performance index variation for this example is $dG(u_b) = dm^f(u_b) - dm_e(u_b)$, and the total variation is $dJ(u_b) = dm_p(u_b)$. Substitution of these relations for $dx_0(t^i)$ and $dx_0(t^f)$, respectively, in Eq. (1) and neglecting the integral of Q in the interval $c_-{}' < t < c_+{}''$ leads to the following

† It has been noted by one of the technical reviewers that the present analysis can be developed using special state variables, instead of special control variables, as configuration parameters. Such an approach is more direct mathematically but, in the author's opinion, identification of configuration adjustment with pseudo-control deflection is preferable from the intuitive viewpoint of a vehicle designer. The resulting tradeoff (sensitivity) relations and numerical procedures are the same for either approach so the choice of viewpoint is open.

‡ In view of the generally pragmatic approach of this analysis, it is preferable to use examples rather than generalized development of this index.

estimate for variation of payload due to a configuration-variable perturbation δu_b:

$$dJ(u_b) = dm_p(u_b) = -dm_e(u_b) - \int_{t^i}^{t^f} \left(\lambda^T \frac{\partial f}{\partial u_b} + \nu^T \frac{\partial \varphi}{\partial u_b} + \mu^T \frac{\partial \eta}{\partial u_b} \right) \delta u_b dt \quad (3)$$

Note the following:

1) The term $dm^f(u_b)$ of dG is eliminated because it is included in the transversality condition for the reference solution.

2) The variation δu_b is fictitious, so the last term of Eq. (2) may be omitted for configuration variables. If u_b is assumed unbounded, $\mu_b = 0$; if u_b is assumed bounded at its nominal value, then $\delta z_b = 0$. This follows from the Euler-Lagrange equation for auxiliary variable z_b.[3]

Further simplification of Eq. (3) results from practical considerations. Most cases of interest do not involve holonomic constraints; hence, $\varphi = 0$. Also, it is unlikely that a primary control variable will be on a limit that simultaneously involves u_b. This is the only situation that would introduce a nonzero value of $\mu^T(\partial \eta/\partial u_b)$. Thus $\nu^T(\partial \varphi/\partial u_b)$ and $\mu^T(\partial \eta/\partial u_b)$ may be deleted from the integrand. These terms may be readily restored when required for an extraordinary problem. With the foregoing conditions, the estimated payload variation due to δu_b reduces to

$$dm_p(u_b) = -dm_e(u_b) - \int_{t^i}^{t^f} \lambda^T \frac{\partial f}{\partial u_b} \delta u_b dt \quad (4)$$

At this point, the integrand may be related to the derivative of the variational Hamiltonian, H. Since $f = \dot{x}$ and $H = \lambda^T \dot{x}$, it follows that

$$\lambda_{ub} = \partial H/\partial u_b = \lambda^T(\partial \dot{x}/\partial u_b) = \lambda^T(\partial f/\partial u_b) \quad (5)$$

The parameter λ_{ub} is the influence function for the particular configuration variable u_b. In terms of the influence function, the payload variation due to δu_b is

$$dm_p(u_b) = -dm_e(u_b) = \int_{t^i}^{t^f} \lambda_{ub} \delta u_b dt \quad (6)$$

In the foregoing development a particular performance index, payload, has been used. As another example, a performance index for maximum range is $X = J \equiv G = R_0(\theta^f - \theta^i)$ or a minimum-fuel problem may use $m_f = J \equiv G = m^i - m^f$. Here, m_f is the mass of primary fuel.

A useful procedure for evaluating optimum-performance tradeoffs for configuration adjustments follows from the development of Eq. (6). The procedure requires integration of an influence function λ_{ub} as defined by Eq. (5). Knowing the sign of λ_{ub}, the desired direction of configuration-variable adjustment $\delta \mu_b$ can be deduced from Eq. (6) or its equivalent. The resulting sign of δu_b is the same as that obtained by treating u_b as a primary control variable and interpreting $\partial H/\partial u_b$ in accordance with the maximum principle.[1]

An auxiliary tradeoff relation may be obtained from Eq. (1). This concerns the effect of terminal-state variations dx_i^i and dx_i^f on the performance index J. By singly considering terminal-value perturbations of state variables in Eq. (1), it follows that $\lambda_i(t^i)$, $\lambda_i(t^f)$, $H(t^i)$, and $H(t^f)$ are influence parameters for the terminal state. The parameters correspond to the partial derivatives below for initial and final conditions, respectively,

$$\begin{aligned} \lambda_i(t^i) &= -\partial J/\partial x_i^i \quad \text{and} \quad H(t^i) = \partial J/\partial t^i \\ \lambda_i(t^f) &= \partial J/\partial x_i^f \quad \text{and} \quad H(t^f) = -\partial J/\partial t^f \end{aligned} \quad (7)$$

These relations apply to neighboring optimum solutions and may be used to estimate the effect upon the performance index of variations in terminal conditions.[2]

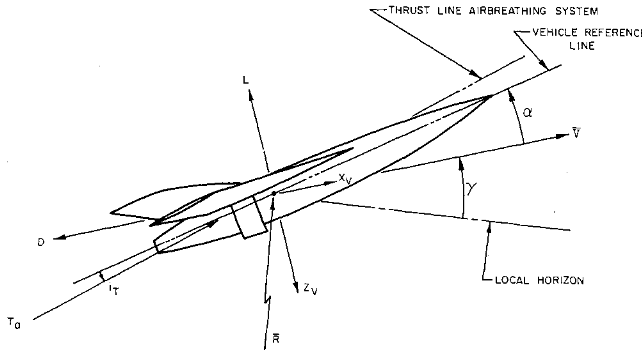

Fig. 1 Vehicle attitude and applied-force components.

Influence Functions

Development of analytical relations for the influence functions λ_{ub} defined by Eq. (5) may be deferred until specific application is determined. The relations presented below are appropriate for the demonstration problems of this paper and generally apply to hypersonic, ramjet-powered aircraft. Vehicle representation and notation are the same as those used in Ref. 3. For convenience, terms of λ_{ub} may be grouped into aerodynamic, propulsion-system, and special categories, as follows:

$$\lambda_{ub} = \partial H/\partial u_b = \Delta\lambda_{ub} \text{ aero} + \Delta\lambda_{ub} \text{ prop} + \Delta\lambda_{ub} \text{ spec} \quad (8)$$

where

$$\Delta\lambda_{ub} \text{ aero} = \frac{qS_w}{m} \left[-\lambda_1 \left(\frac{\partial C_d}{\partial u_b} + \frac{C_D}{S_w} \frac{\partial S_w}{\partial u_b} \right) + \frac{\lambda_2}{V} \left(\frac{\partial C_L}{\partial u_b} + \frac{C_L}{S_w} \frac{\partial S_w}{\partial u_b} \right) \right] \quad (9)$$

$$\Delta\lambda_{ub} \text{ prop} = \frac{T_a}{m} \left[\left(\frac{1}{C_F} \frac{\partial C_F}{\partial u_b} + \frac{1}{A_c} \frac{\partial A_c}{\partial u_b} \right) \times \left(\lambda_1 \cos(\alpha + i_T) + \frac{\lambda_2}{V} \sin(\alpha + i_T) - \lambda_5 \frac{m}{g_0 I_{sp}} \right) + \lambda_5 \frac{m}{g_0 I_{sp}{}^2} \frac{\partial I_{sp}}{\partial u_b} \right] \quad (10)$$

and for the special case $u_b = i_T$,

$$\Delta\lambda_{ub} \text{ spec} = \Delta\lambda_{iT} = \frac{T_a}{m} \times \left(-\lambda_1 \sin(\alpha + i_T) + \frac{\lambda_2}{V} \cos(\alpha + i_T) \right) \quad (11)$$

As has been mentioned earlier, u_b may be one of a variety of configuration features or design parameters. A specific influence function may be evaluated by substituting values for the applicable configuration-data derivatives, together with the multiplier values and flight-path conditions from the reference optimum solution, in the foregoing relations. In this procedure, the computation is simple and is separate from the reference solution. Hence, many configuration variables may be evaluated and ranked with respect to performance tradeoffs, even though the optimum flight path is predominantly dynamic. Sensitivity of performance to design variables can be readily assessed, and thus critical configuration variables may be identified early in the design process. Values of configuration-data derivatives ($\partial C_D/\partial u_b$, $\partial C_F/\partial u_b$, etc.) based upon the judgment of qualified specialists are adequate for many useful results.

Unit Perturbations

Evaluation of an influence function λ_{ub} using Eq. (8) requires that configuration-data derivatives be related to an appropriate unit of the configuration variable u_b. Experience and judgment are important factors in selecting a value for the unit perturbation δu_b. For example, suppose u_b is wing-thickness ratio t/c and its nominal value is $t/c = 0.04$. Then a reasonable unit perturbation would be $\delta(t/c) = 0.002$. Such a decision is necessary before influence functions are used for comparing several configuration variables or before determining quantitative performance tradeoffs.

Application of the Analysis—Hypersonic Aircraft

The foregoing analysis is directed toward the use of simple computation procedures together with the judgment of technical experts in a vehicle design situation. Unique and timely information is made available to the vehicle designer, and the utility of the variational analysis is increased. Since approximations and judgments are involved, it is important to establish range of applicability and limitations of the analysis. This is best accomplished by considering relevant examples to test and demonstrate application of the analysis. Primary motivation of the present work is the development of design methods for hypersonic aircraft. Thus, examples used involve this class of vehicles. The methods and techniques of the present analysis, however, are directly applicable to the whole spectrum of aerodynamic vehicles, from subsonic turbine-powered aircraft to rocket-propelled spacecraft.

Three representative examples are summarized below. Additional results and solution details are given in Ref. 1. Vehicle representation along with nominal aerodynamic and propulsive-system data are shown in Figs. 1–3.

Quasi-Steady Flight—Maximum Range

As an exploratory case for demonstrating the use of influence functions, a segment of quasi-steady flight at $M = 10$ was selected. This limited example permits several configuration variables to be investigated, singly and in combination, with a modest computing effort. Thus the validity and limitations of the analysis were checked before more comprehensive problems were investigated. The procedure in this case was to 1) obtain a reference optimum solution, 2) compute influence functions and estimate performance tradeoffs, and 3) verify each result with a corresponding variational solution.

Simply stated, in this example it is desired to achieve maximum range for a given expenditure of fuel. Range is the performance index, and the path is restricted to quasi-steady flight at $M = 10$. This condition occurs for the nominal configuration with initial state values: $V^i = 10{,}496$ fps, $\gamma^i = 0$, $h^i = 124{,}000$ ft, and $m^i = 200{,}000$ lb. Satisfying transversality criteria for maximum range (range multiplier, $\lambda_4{}^f = -1.0$, and variational Hamiltonian $H^f = 0$) with $V^f = V^i$, $\gamma^f = \gamma^i$, and $h^f = h^i$ provides the reference optimum solution. The resultant range is 1365 naut miles for 16,500 lb of fuel. Angle of attack remains nearly constant throughout the flight ($\alpha = 3.70° \pm 0.036°$).

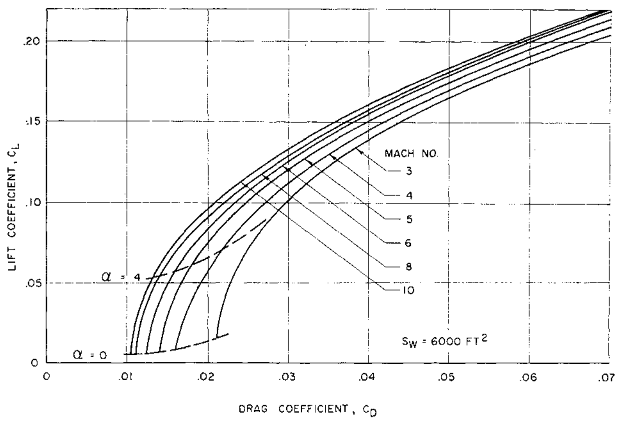

Fig. 2 Aerodynamic force coefficient data.

Results of configuration-variable perturbations are summarized in Table 1. Influence functions λ_{u_b} were evaluated by using Eqs. (8–11). Estimated range increments were computed using the reference solution. Structural effects were deferred by assuming $dm_e = 0$, and the approximation $dX(u_b) = -\lambda_{u_b}$ (av) $(t^f - t^i)\delta u_b$ was used instead of the integral equivalent of Eq. (6). Simple calculation is possible in this case, since multiplier histories $\lambda_i(t)$ for the reference solution either vary linearly or remain essentially constant. The actual increments are the differences between range of the reference solution and that of the optimum flight path for the corresponding configuration-variable perturbation.

These results demonstrate a reliable relationship between the tradeoff estimated using the influence function and the actual performance tradeoff. It is evident, and not unexpected, that reduced accuracy accompanies large perturbations of configuration variables or large changes in the performance index, e.g., A_c. Likewise, reliability of the estimate deteriorates when the tradeoff is small. When an influence function λ_{u_b} is small, it suggests that either the configuration variable u_b is not critical or the nominal value of u_b is essentially the optimum. (Although not included in Table 1, interpretation of the thrust incidence influence function λ_{i_T} was inconclusive. Additional optimum solutions were obtained for $i_T = 2°$, $-2°$, and $-4°$ which verified the optimality of the nominal value, $i_T = 0$. A reliable and conclusive interpretation of λ_{i_T} resulted from each of the additional solutions.[1])

Table 1 Range tradeoff summary, quasi-steady flight at $M = 10$

Nominal design variable u^b	Configuration[b] characteristic increment(s)	Range increment,[a] naut miles estimated[c]	actual[d]
Wing thickness ratio	$\Delta C_D = -0.0001$	50	43
Forebody overhang	$\Delta C_L = 0.0005\ \alpha$/deg	55	55
Forebody fineness ratio	$\Delta C_D = -0.0001$; $\Delta C_L = 0.0005\ \alpha$/deg	104.5	98.5
Nozzle exit velocity ratio	$\Delta C_F = 0.002$	14.5	13.5
	$\Delta I^4_{sp} = 5\ \alpha$ sec/deg	12	13
	$\Delta C_F = 0.002$; $\Delta I^4_{sp} = 5\ \alpha$ sec/deg	26	27
Inlet capture area	$\Delta A_C = 2$ ft²	104	62.5
Wing-body-engine arrangement	$\Delta C_F = 0.002$; $\Delta C_L = 0.0005\ \alpha$/deg	69	70

[a] Nominal range is 1365 naut miles.
[b] $\Delta C_D = (\partial C_D/\partial u_b)\delta u_b$ etc.
[c] Estimate based upon integration of influence function.
[d] Result obtained from corresponding optimum solution.

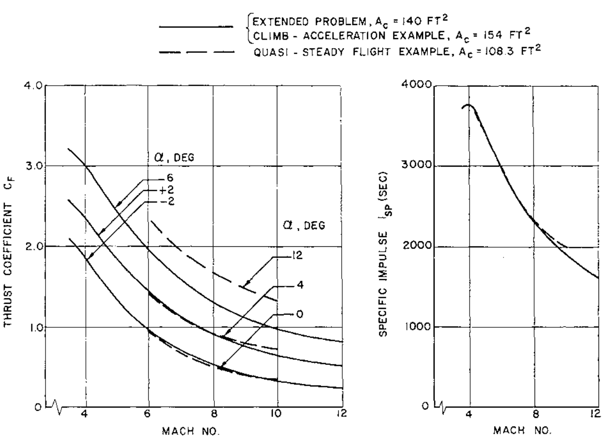

Fig. 3 Propulsion system characteristics.

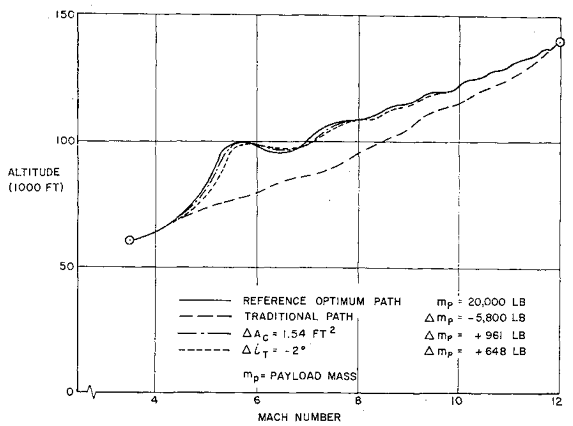

Fig. 4 Altitude-Mach-number profile; climb-acceleration example.

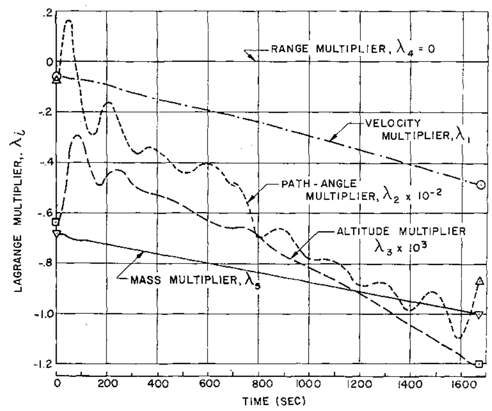

Fig. 6 Lagrange multiplier histories; climb-acceleration example.

Identification of specific configuration variables in Table 1 was done a posteriori. The original purpose was to explore sensitivity of the various terms in Eqs. (9–11). This procedure is acceptable for the present example and demonstrates a utility of the analysis that provides useful results even without prior detailed determination of configuration-variable perturbation effects on the aerodynamic and propulsion-system characteristics.

Climb-Acceleration—Maximum Payload

The second example demonstrates application of the analysis to a climb-acceleration flight-path segment and involves the extended-problem procedure. This is a key example because nonsteady flight-path conditions lead to influence functions that vary significantly. In this case, an optimum path is to be reckoned with respect to maximum payload at a given final condition: $h^f = 140{,}000$ ft, $M^f = 12$ ($V^f = 12{,}934$ fps), and $\gamma^f = 0$. Initial conditions are $h^i = 60{,}000$ ft, $M^i = 3.5$ ($V^i = 3388.3$ fps), $\gamma^i = 4°$, and $m^i = 230{,}000$ lb. The transversality condition, for maximum payload and the given terminal state, requires that 1) $\lambda_4{}^f = 0$ (range multiplier), 2) $\lambda_5{}^f = -1.0$ (mass multiplier), and 3) $H^f = 0$ (variational Hamiltonian). Flight profiles for the reference optimum solution are shown in Figs. 4 and 5. The multiplier histories are shown in Fig. 6. For the reference solution, $A_c = 154$ ft², $i_T = 6°$, and the nominal payload is 20,000 lb. It was convenient to use state values at $M = 10.0$ as the intermediate terminal condition for the first segment of the extended optimum flight path (see Appendix). The slight oscillatory character of the flight path is a typical and expected motion of an aircraft operated at fixed throttle.

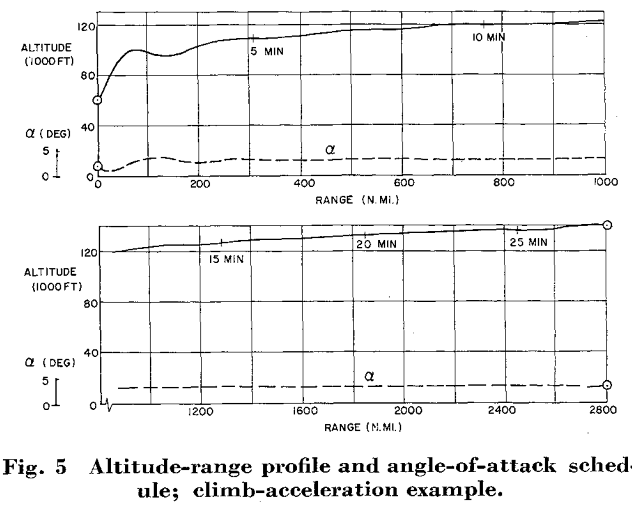

Fig. 5 Altitude-range profile and angle-of-attack schedule; climb-acceleration example.

The gain in payload resulting from following an optimum-flight path was determined using the traditional h-M profile shown in Fig. 4. The optimum path achieves the same final-state values with 5800 lb more payload. Additional performance gains are likely because of less severe flight conditions along the optimum path.

In this example, performance tradeoffs were evaluated for two configuration variables. Engine size, as related to capture area, was considered first (i.e., $u_b = A_c$). The derivatives of C_F and I_{sp} are assumed to be zero, and aerodynamic characteristics are assumed to be independent of A_c. These assumptions are expedient for the example only. For this case the influence function λ_{A_c}, as developed from Eqs. (8–10), is

$$\lambda_{A_c} = \frac{T_a}{mA_c}\left[\lambda_1 \cos(\alpha + i_T) + \frac{\lambda_2}{V}\sin(\alpha + i_T) - \lambda_5 \frac{m}{g_0 I_{sp}}\right] \tag{12}$$

Simple hand calculations were used to compute values of λ_{A_c} at points along the reference solution. These values are shown in Fig. 7. The negative λ_{A_c} values suggest that 1) payload may be increased by increasing A_c and that 2) the value of A_c is more important during the later (high-speed) portion of the flight.

Computation of the payload tradeoff for a change δA_c requires evaluation of Eq. (6) with $u_b = A_c$. The change in structure $dm_e(A_c)$ is determined independently.§ Here it is convenient to let $dm_e = 0$ and focus attention on the integral

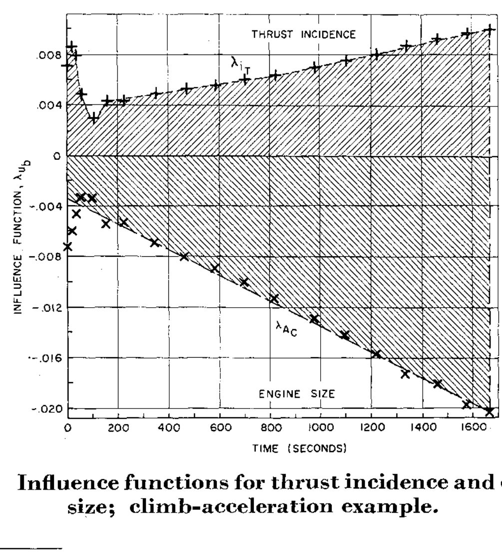

Fig. 7 Influence functions for thrust incidence and engine size; climb-acceleration example.

§ As part of the design team the weight analyst and structural designer would provide values at this point of the design process.

Table 2 Tabulated terminal values for climb-accelerate example

	Segment I		Segment II		
	Initial value, $M = 3.5$	Intermediate value, $M = 10.0$	Intermediate value, $M = 10.0$	Final value, $M = 12.0$	Desired final value
State variables					
V, fps	3,388.3	10,449.5	10,449.5	12,933.8	12,934 ± 10
γ, deg	4.0	0.36	0.36	−0.11	0 ± 0.15
h, ft	60,000	121,852	121,852	139,715	140,000 ± 500
θ, deg	0	15.355	15.355	46.698	free
m, lb	230,000	195,093	195,093	165,175	free
t, sec	0	699.05	699.05	1,668.68	free
Variational parameters					
$\lambda_1 \times 10$	−0.57235	−2.16873	−2.16829	−4.82246	...
$\lambda_2 \times 10^{-1}$	−0.72369	−4.93539	−4.86106	−8.68019	...
$\lambda_3 \times 10^3$	−0.64351	−0.66348	−0.66268	−1.19690	...
λ_4	0	0	0	0	0
λ_5	−0.68795	−0.82067	−0.82067	−1.000	−1.0
H	−0.01002	0.01135	0.01135	0.01135	0 ± .02

term that estimates payload change for flight along a neighboring optimum path compatible with the change in A_c. Again, simple computation of the area bounded by the dashed line in Fig. 7 provides a useful result. The simply estimated increase in payload is 980 lb for a 1% increase in $A_c(\delta A_c = 1.54\ \text{ft}^2)$. Flight along an optimum path with $A_c = 155.54$ ft^2 resulted in an actual payload increase of 961 lb. The resulting h-M flight profile is included in Fig. 4.

The second configuration-variable tradeoff evaluated in this example is the net-thrust incidence angle, i.e., $u_b = i_T$. Using Eq. (11) the influence function λ_{i_T} was readily evaluated as shown in Fig. 7. It is evident from the sign of λ_{i_T} that i_T should be reduced. The estimated payload tradeoff for $\delta i_T = -2°$ is 730 lb. The payload increment for flight along an optimum path with $i_T = 4°$ was 648 lb. Accuracy of this estimate is somewhat less than that of the preceding case; however, an increment of $-2°$ represents a relatively large configuration-variable adjustment.

Terminal values for the segments of this problem are listed in Table 2. These values are shown to illustrate 1) the precision used to satisfy final conditions and 2) the general magnitude of multiplier discontinuities at the extended-problem juncture.

Extended Problem—Maximum Range

A final example is reviewed briefly to further demonstrate extended-problem procedure and to illustrate use of the analysis to evaluate operational procedures.¶ In this example, maximum range was to be achieved along a flight path that includes descent-deceleration. Thus it is necessary to prescribe a nominal operating procedure that causes the vehicle to decelerate. The reference solution can then be used to evaluate both configuration variables and operating procedure.

Initial state of the preceding example was used. Variational criteria for a maximum-range solution were the same as those of the first example. The nominal operating procedure was a climb-acceleration at full throttle until $m' = 160{,}000$ lb (approximate end of cruise condition), followed by a descent-deceleration to the final condition ($M^f = 3.5$) at 25% of full throttle. A range of 6620 naut miles results for the reference solution with $i_T = 6°$ and $A_c = 140$ ft^2. The h-M schedule is shown in Fig. 8. Operating procedure adjustments considered in this example are 1) the fuel assigned to climb-acceleration and 2) the thrust level during descent-deceleration.

The net-range tradeoff due to adjusting fuel assigned to climb-acceleration may be evaluated from the combined effect of terminal-mass perturbation $\delta m'$ at the climb-descent juncture. Using Eq. (1) to evaluate changes in vehicle mass at this operational juncture results in the following tradeoff factors:

$$dX/dm' = -0.0805 \text{ naut mile/lb at end of climb-acceleration}$$

$$dX/dm' = +0.264 \text{ naut mile/lb at start of descent-deceleration}$$

These values indicate that for each mile the climb segment is shortened, the descent segment gains approximately 3 miles. Thus m' should be increased (climb fuel reduced) to effect a net gain in over-all range.

The range tradeoff for adjustment of thrust level during descent involves an influence function. A parameter τ, defined by the relation $T_a = (\tau/100)C_F q A_c$ $(0 \leq \tau \leq 100)$, may be considered as a configuration variable and the preceding method applied to estimate the effect of thrust-level adjustment during descent. The influence function for thrust level, developed from Eq. (8), is

$$\lambda_\tau = \frac{C_F q A_c}{100\, m}\left[\lambda_1 \cos(\alpha + i_T) + \frac{\lambda_2}{V}\sin(\alpha + i_T) - \lambda_5 \frac{m}{g_0 I_{sp}}\right] \qquad (13)$$

This function was evaluated along the reference flight path and is shown in Fig. 9. Integration of λ_τ over the descent segment of the flight path provides a quantitative tradeoff for adjustment of the thrust level during descent. The re-

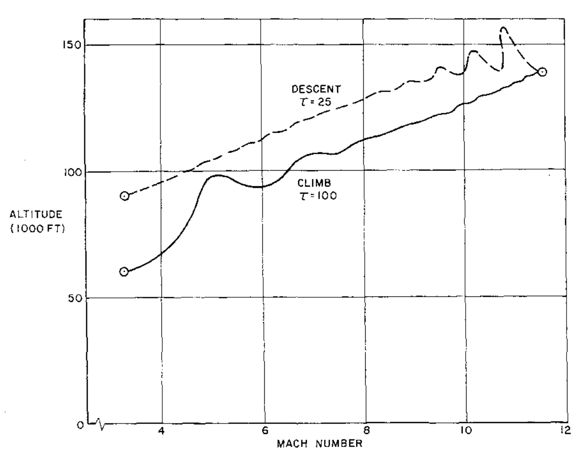

Fig. 8 Altitude-Mach-number profile; extended problem.

¶ This problem is discussed more fully in Ref. 1.

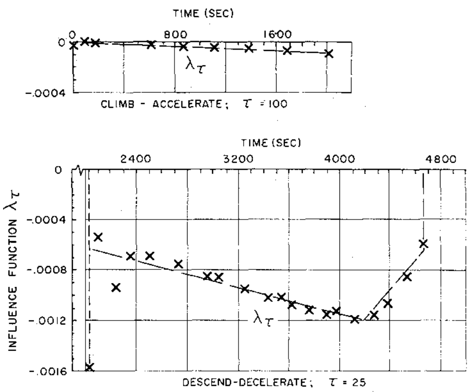

Fig. 9 Thrust-level influence function; extended problem.

sulting tradeoff parameter is $dX/d\tau = 83.6$ naut miles/1.0. Thus it is suggested that more thrust should be used during descent to increase the over-all range.

The present example illustrates the initial step in the development of an optimum configuration and operating procedure for a long-range hypersonic aircraft. Since λ_τ has a relatively small, negative value during climb-acceleration, it may be concluded that engine size is appropriate for this mission and that full throttle operation should be continued until start of descent. A positive value of λ_τ would signal the start of part-throttle operation (cruising flight). Finally, the relatively large magnitude of λ_τ during descent indicates that range performance is particularly sensitive to the thrust level along the descent-deceleration path.

As an example of the combined effect of these adjustments, consider an increase in thrust level during descent from $\tau = 25$ to $\tau = 27.5$ along with a compensating reduction in fuel for climb by increasing m' to 162,000 lb ($\delta m' = 2000$ lb). Conservative extrapolation of the foregoing tradeoff parameter predicts a net range increase of 500 naut miles for such an operating procedure adjustment. This estimated gain would be for an optimum path having the same terminal conditions as the reference solution.

Discussion

The analysis and results presented in this report are the consequence of an effort to develop useful design information from the additional variables that accompany application of variational calculus to aerospacecraft performance optimization. A primary goal has been to provide timely and practical inputs for the aerospacecraft design process. For this reason simple relations, slide-rule computations, and qualified technical judgments are used whenever possible. In addition, a solution technique is demonstrated that reduces inherent numerical difficulties of comprehensive variational procedures. Extension of the analysis and examples cited to other types of aerospacecraft is straightforward.

Use of design variables as a special class of control variables is akin to parametric studies of "rubber" vehicle configurations. A basic question arises in parameteric treatment of multivariable problems having path-dependent functions. Performance evaluated along a prescribed flight path or control schedule may be significantly different from the performance using an appropriate optimum flight path. Such a situation is apt to occur whenever the optimum flight path is sensitive to a particular design variable; this sensitivity is related to identification of critical design variables.

A characteristic behavior of optimum flight paths is evident from present and prior performance optimizations.[1,3] This behavior pertains to extended problems where an intermediate section of the path is practically insensitive to terminal-state values (see Appendix). Generally, three sectors of an extended flight path may be identified.

1) The starting sector is a transition flight path from the initial state to the intermediate sector. This path is a strong function of initial values.

2) The intermediate sector is a relatively stable flight path that remains essentially invariant and thus independent of the terminal state. This sector of an optimum path, along with forward and rearward extensions, might be termed a "natural" optimum path.

3) The concluding sector is a transition from the intermediate sector to the required final state. The concluding sector is a strong function of final values.

Each of the preceding sectors may be identified in the climb-acceleration example. With more complex problems involving operating-procedure adjustments, transitional sectors may occur with each adjustment. In the foregoing extended problem, a transition flight path is found at the start of climb and at the start of descent.

Generally intuitive and pragmatic arguments have been used in the present analysis. Such an approach, coupled with the use of simple inputs and computation procedures, requires that each class of problem or new configuration be investigated carefully. This caution precludes exceeding configuration-variable perturbation limits and assures validity of conclusions. Perturbation limits may be related to the unit variations appropriate for influence functions and tradeoff evaluation.

Use of simple computation procedures and technical judgment inputs introduces a limitation of the analysis that is often an advantage. Whenever the estimated tradeoff is very small relative to the reference solution performance index, results are inconclusive. Within this "mud level" of uncertain results, it may be concluded that either performance is not critically dependent on the particular design variable or that the nominal value is essentially an optimum value. For either conclusion, the design process gains because available resources can be directed toward improvement of more critical configuration features.

In the present analysis, the influence function and tradeoff computations are separate from the computation of the reference solution. Hence, having once obtained the reference flight path, new and varied combinations of configuration variables may be investigated promptly without further optimum flight-path solutions.

Several illustrative tradeoff parameters were developed in the first example problem without detail analysis of configuration-variable effects on aerodynamic and propulsive-system characteristics. Actually, the configuration variables were not specifically identified except to suggest parameters compatible with the derivative and perturbation values used. Configuration-characteristic derivatives are not inputs for the reference optimum solution; hence, they may be evaluated independently to an appropriate degree of accuracy and completeness. For example, the aerodynamic derivatives $\partial C_D/\partial u_b$ and $\partial C_L/\partial u_b$ may be the result of an elaborate study and test program, or they may be approximate values casually assumed to assess sensitivity of the performance index. Thus the analysis may be readily adapted to available data and to the design situation.

Experience in application of these variational techniques is a principal factor in reducing time and effort required to achieve a specific result. In general, it is advisable to build up experience with representative problems for each class of vehicles. Thereafter, solutions and particular results can be obtained rather quickly for a variety of related problems. Accumulated experience with the preceding examples contributed significantly to solution of a climb-acceleration problem for an entirely new hypersonic vehicle. Solution of this related problem was accomplished with approximately one-eighth of the effort used to obtain the reference solution for the second example of the present paper.

Although flight-path optimization per se was not the primary purpose of the present work, the second representative example emphasizes the importance of using methods and criteria of variational calculus in performance analysis and flight-path development of advanced aerodynamic vehicles.

Conclusions

General conclusions based on the foregoing analysis and examples are:

1) Influence functions and performance tradeoff parameters provide timely and reliable evaluation of various configuration variables along a complex flight profile. Simple computations are generally adequate.

2) The net effect on performance of configuration variation, particularly that due to adjustment of the optimum flight path, may be readily evaluated.

3) Performance of advanced aerodynamic vehicles is often critically dependent on the flight path, thus requiring continuous dynamic flight-path analysis and application of comprehensive optimization techniques.

4) Experience continues to be an important factor in the application of variational methods to aerospacecraft performance analysis and flight-path optimization.

Appendix: Extended-Problem Procedure

A practical difficulty of applied variational analysis occurs when a solution must be extended to a point such that the numerical accuracy required of the unknown initial values exceeds the available (or a reasonable) number of significant figures. If desired final conditions have not yet been achieved, the problem will either have to be revised to remain within range of the numerical procedure or abandoned in favor of a less comprehensive method of analysis. This perversity has been a principal deterrent to use of variational techniques in complex problems. Fortunately, in many problems of practical importance, this difficulty can be alleviated without compromising the analysis or the problem.

Experience with a variety of performance-optimization problems[1,3] has shown that starting and intermediate portions of a solution become essentially invariant before the problem reaches the above point of numerical insolvency. Control and multiplier histories show a similar behavior. Thus when it is established that the solution of a problem becomes essentially invariant over a significant portion of the flight path, the problem may be termed an "extended" problem.

A practical solution procedure follows from this concept of an extended problem. Once it is apparent that the flight path, control schedules, and multiplier histories have converged over initial and intermediate sectors of the solution, there is no need to recompute those parts of the path for each trial integration into the final phase of the flight. (Here, convergence means that to the usual scale of plotting or displaying these functions the differences between neighboring paths become indistinguishable.)

Reduction of numerical-accuracy requirements and machine computing time has been achieved by 1) accepting the converged part of the solution, 2) selecting a convenient point on this path as the initial state for restarting the solution, and 3) continuing the solution to the desired final condition.

The juncture selected in step 2 becomes a special corner, at which the variables λ_i and H may have a discontinuity without compromising the corner condition,[2,3] since the $dx_i(t_c)$ and dt_c are all zero. This technique achieves a useful solution to an extended problem with a nominal numerical procedure. A similar argument applies at special junctures where a change in operating procedure must be made. For example, the point between a cruise-flight segment and a descent segment requires state variables to be continuous while multipliers are allowed to be discontinuous.

Since variational analysis provides necessary conditions (not sufficient) for an optimum, multiple solutions may exist and do in fact occur. In using the extended problem procedure, as in solving any variational problem, eash solution should be examined critically to assure that the proper solution is found. Engineering judgment is usually adequate to identify and discard any lesser optimum solution that may be obtained. Large fluctuations of flight-path characteristics and multiplier histories are the principal basis for such judgments.

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